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ON WEAKLY NULL FDD'S IN BANACH SPACES

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ABSTRACT

In this paper we show that every sequence (F_n) of finite dimensional subspaces of a real or complex Banach space with increasing dimensions can be "refined" to yield an F.D.D. (G_n), still having increasing dimensions, so that either every bounded sequence (x_n), with $x_n \in G_n$ for $n \in \mathbb{N}$, is weakly null, or every normalized sequence (x_n), with $x_n \in G_n$ for $n \in \mathbb{N}$, is equivalent to the unit vector basis of ℓ_1.

Crucial to the proof are two stabilization results concerning Lipschitz functions on finite dimensional normed spaces. These results also lead to other applications. We show, for example, that every infinite dimensional Banach space X contains an F.D.D. (F_n), with $\lim_{n\to\infty} \dim(F_n) = \infty$, so that all normalized sequences (x_n), with $x_n \in F_n$, $n \in \mathbb{N}$, have the same spreading model over X. This spreading model must necessarily be 1-unconditional over X.

1. Introduction

Let (F_n) and (G_n) be two sequences of finite dimensional subspaces of a Banach space X. We say (F_n) is **large** if $\lim_{n\to\infty} \dim F_n = \infty$. We say (G_n) is a **refinement** of (F_n) if there is a strictly increasing sequence $(k_n) \subset \mathbb{N}$ so that G_n is a subspace of F_{k_n} for all $n \in \mathbb{N}$. If each (F_n) has a given basis $\mathbf{b}_n = (f_i^{(n)}: 1 \le i \le \dim F_n)$, we say (G_n) is a **block refinement** of (F_n) with respect to $(\mathbf{b}_n)$ if for some (k_n) as above, G_n is spanned by a block basis of $\mathbf{b}_{k_n}$ for all n. (F_n) is

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called an **F.D.D. (Finite Dimensional Decomposition)** if (F_n) is a Schauder-decomposition for its closed linear span. It is readily seen (using the standard Mazur argument) that every large sequence (F_n) has a large F.D.D. refinement (G_n); moreover (G_n) can be chosen to be a block-refinement of (F_n) with respect to $(\mathbf{b}_n)$ for a given sequence of bases $(\mathbf{b}_n)$ of the F.D.D. We say (G_n) is **weakly null** if every bounded sequence (x_n) with $(x_n) \in G_n$ for all n, is weakly null. We say (G_n) is **uniformly-ℓ_1** if there exists a $C > 0$ such that all normalized sequences (x_n) with $x_n \in G_n$ for all n, are C-equivalent to the unit vector basis of ℓ_1. Of course (G_n) is uniformly-ℓ_1 precisely when (G_n) is an ℓ_1-F.D.D.; that is, the closed linear span of the G_n's is canonically isomorphic to $(\sum \oplus G_n)_1$, the space of all sequences (g_n) with $g_n \in G_n$ for all n and $\|(g_n)\| \stackrel{\text{df}}{=} \sum \|g_n\| < \infty$.

Except as noted, our terminology is standard and may be found in the book [LT]. All Banach spaces are assumed to be separable.

If (x_n) (resp. (G_n)) is a (finite or infinite) sequence of elements of (resp. finite-dimensional subspaces of) a Banach space X, $[x_n]$ (resp. $[G_n]$) denotes the closed linear span of (x_n) (resp. (G_n)). S_X denotes the unit sphere of X and $\text{Ba}(X)$ its unit ball.

Our main result is the following.

THEOREM 1: *Let (F_n) be a large sequence of finite-dimensional subspaces of a Banach space X. Then there exists a large refinement (G_n) of (F_n) so that either (G_n) is a weakly null* FDD *or (G_n) is an ℓ_1-*FDD. *Furthermore if there is a given sequence $(\mathbf{b}_n)$ of bases of the F_n's with uniformly bounded basis constants, then the above sequence (G_n) can be chosen to be a block refinement of (F_n) with respect to $(\mathbf{b}_n)$.*

Theorem 1 can be viewed as a block version of the ℓ_1-theorem of the second named author, which says that every normalized sequence (x_n) in a Banach space X has a subsequence which is either equivalent to the unit vector basis of ℓ_1 or is weak Cauchy [R1]. Using Krivine's theorem [K] (which is also used the in proof of Theorem 1), one gets further structural consequences of this block version. Krivine's theorem (as refined in [R2] and finally in [L]) may be formulated as follows:

Given a large sequence (F_n) of finite-dimensional subspaces of a Banach space with bases $(\mathbf{f}_n)$ with uniformly bounded basis constants, there exists a block refinement (G_n) of (F_n) with block bases $(\mathbf{g}_n)$ of the $\mathbf{f}_n$'s and a $1 \le p \le \infty$ so *that for all n, $n = \dim(G_n)$ and $\mathbf{g}_n$ is $1 + \frac{1}{n}$-equivalent to the unit vector basis of ℓ_p^n.*

Of course it thus follows that the G_n's in the conclusion of Theorem 1 can be chosen to be uniformly isomorphic to ℓ_p^n, for some $1 \leq p \leq \infty$. We thus obtain immediately the following result.

COROLLARY 2: *Let (F_n) be a large sequence of finite dimensional subspaces of a Banach space X, with given bases $(\mathbf{b}_n)$ with uniformly bounded basis constants; and assume no normalized sequence (f_n) with $f_n \in F_n$ for all n, has a weak Cauchy subsequence. Then there exists $1 \leq p \leq \infty$ and a block refinement (G_n) of (F_n) with respect to $(\mathbf{b}_n)$, such that $[G_n]$ is canonically isomorphic to $(\sum \oplus \ell_p^n)_1$.*

Now Corollary 2 trivially implies that *if X has the Schur property and contains ℓ_p^n's uniformly, then $(\oplus \ell_p^n)_1$ embeds in X*. Of course this is trivial if $1 \leq p \leq 2$, since then ℓ_p is finitely represented in ℓ_1. However the following immediate block version does not appear to be obvious for any value of p larger than 1.

COROLLARY 3: *Let X have the Schur property, and suppose, for some $1 < p \leq \infty$, that ℓ_p is block finitely represented in a particular basic sequence (x_j) in X. Then some block basis of (x_j) is equivalent to the natural basis of $(\sum \oplus \ell_p^n)_1$.*

A famous question in Banach space theory was whether any infinite dimensional Banach space X which does not contain ℓ_1 isomorphically must contain an infinite-dimensional subspace with a separable dual. This is equivalent to asking whether such an X contains a shrinking basic sequence (x_n); i.e., a basic sequence (x_n) so that each bounded block basis (y_n) is weakly null. Of course if (x_n) is such a sequence and (k_n) is an increasing sequence in $\mathbb{N} \cup \{0\}$ with $k_{n+1} - k_n \to \infty$, then setting $F_n = [x_i]_{i=k_n+1}^{k_{n+1}}$, (F_n) is a large weakly null FDD. However T. Gowers [G2] has recently solved the general problem in the negative; i.e., there is a Banach space X not containing ℓ_1, with *no* shrinking basic sequences. Nevertheless, Theorem 1 gives at once that *every basic sequence in any X not containing ℓ_1 has a block basis (x_n) which yields large weakly null FDD's as above.*

COROLLARY 4: *If ℓ_1 is not isomorphically contained in X and (y_n) is a basic sequence in X, then given an increasing sequence $(k_n) \subset \mathbb{N} \cup \{0\}$ with $k_{n+1} - k_n \to$*

∞, there exists a block basis (x_n) of (y_n) so that (F_n) is weakly null, where $F_n = [x_i]_{i=k_n+1}^{k_{n+1}}$ for all n.

Another corollary of Theorem 1 is the following result, stated in [R6, Corollary 22] and proved there using Theorem 1 and the Borsuk antipodal mapping theorem. Corollary 5 was obtained independently by W. B. Johnson and T. Gamelin [CGJ].

COROLLARY 5: *Assume ℓ_1 does not embed in X, where X is an infinite dimensional Banach space. Then there exists a normalized weakly null basic sequence (x_i) in X possessing a normalized sequence of biorthogonal functionals.*

The main tools needed to prove Theorem 1 will be the following two finite dimensional "stabilization principles." The first one was observed by V. Milman (see [MS, p.6]) in connection with A. Dvoretzky's famous theorem that in every infinite dimensional Banach space one finds, for each $\varepsilon > 0$ and $n \in \mathbb{N}$, an n-dimensional subspace F which is $(1+\varepsilon)$-isomorphic to ℓ_2^n. The second stabilization principle follows mainly from Lemberg's [L] proof of Krivine's theorem.

FIRST STABILIZATION PRINCIPLE: *For every $C > 0$, $\varepsilon > 0$ and $k \in \mathbb{N}$ there is an $n = n(C, \varepsilon, k) \in \mathbb{N}$ so that: If F is an n-dimensional normed space and $f\colon F \to \mathbb{R}$ is C-Lipschitz (i.e., $|f(x) - f(y)| \le C\|x - y\|$ for $x, y \in F$), then there is a k-dimensional subspace G of F so that*

$$\operatorname{osc}(f|_{S_G}) \equiv \sup\{|f(x) - f(y)|: x, y \in S_G\} < \varepsilon .$$

SECOND STABILIZATION PRINCIPLE: *For all $C > 0$, $\varepsilon > 0$ and $k \in \mathbb{N}$ there is an $n = n(C, \varepsilon, k) \in \mathbb{N}$ so that if F is an n-dimensional normed space with a basis $(x_i)_{i=1}^n$, whose basis constant does not exceed C, and if $f\colon F \to \mathbb{R}$ is C-Lipschitz, then there is a block basis $(y_i)_{i=1}^k$ of $(x_i)_{i=1}^n$ so that*

$$\operatorname{osc}\Big(f|_{S_{[y_i]_{i=1}^k}}\Big) < \varepsilon .$$

Since on the one hand the second stabilization principle nearly follows in a straightforward manner from the proof of Krivine's theorem (the only exception is the case $F = \ell_\infty^n$), but on the other hand does not follow from the statement of Krivine's theorem itself, we will sketch the proof in section 3.

The next result gives another application of the above stabilization principles. The result yields that for a given Lipschitz function f and large sequence (F_n)

of X of finite-dimensional subspaces, there exists a large refinement (G_n), a Banach space E with a one-unconditional basis (e_j), and a function $\tilde{f}$: $E \to \mathbb{R}$ so that for all sequences (x_i) with $x_i \in S_{G_i}$ for all i, and all k, and all sequences $(\alpha_i) \in \mathrm{Ba}(\ell_\infty)$

$$\tilde{f}\Big(\sum_{i=1}^k \alpha_i e_i\Big) = \lim_{n_k > \cdots > n_1 \to \infty} f\Big(\sum_{i=1}^k \alpha_i x_{n_i}\Big) .$$

The result may be formulated quantitatively as follows: (c_{00} denotes the linear space of finitely supported real valued functions on $\mathbb{N}$. We write for $A, B \in \mathbb{R}$ and $\varepsilon > 0$, $A \overset{\varepsilon}{=} B$ if $|A - B| < \varepsilon$.)

THEOREM 6: *Let X be an infinite dimensional Banach space and let f: $X \to \mathbb{R}$ be Lipschitz. Let $(\varepsilon_n) \subset \mathbb{R}_+$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and let (F_n) be a large sequence of finite dimensional subspaces of X. There exists a large refinement (G_n) of (F_n) and a function $\tilde{f}$: $c_{00} \cap \mathrm{Ba}(\ell_\infty) \to \mathbb{R}$ so that: For all $k \in \mathbb{N}$ and $n_1, n_2, \ldots, n_k \in \mathbb{N}$ with $k \le n_1 < n_2 < \cdots < n_k$, and all $(\alpha_i)_{i=1}^k \in \mathrm{Ba}(\ell_\infty^k)$,*

$$\tilde{f}(\alpha_1, \alpha_2, \ldots, \alpha_k, 0, 0, \ldots) \overset{\varepsilon_k}{=} f\Big(\sum_{i=1}^k \alpha_i x_i\Big)$$

whenever $x_i \in S_{G_{n_i}}$ for $1 \le i \le k$. Moreover if each F_n has a given basis $\mathbf{b}_n$ whose basis constant does not exceed some fixed number, (G_n) may be chosen to be a block refinement of (F_n) with respect to $(\mathbf{b}_n)$.

Theorem 6 has a consequence concerning spreading models, and in fact the Banach space "E" given in the above qualitative formulation may be chosen to be a spreading model of X. Recall that (see e.g., [BL] or [O]) every seminormalized basic sequence in X admits a subsequence (x_n) satisfying: For all $x \in X$, $k \in \mathbb{N}$ and $(\alpha_i)_{i=1}^k \subseteq \mathbb{R}$,

$$\lim_{n_1\to\infty} \lim_{n_2\to\infty} \cdots \lim_{n_k\to\infty} \Big\| x + \sum_{i=1}^k a_i x_{n_i} \Big\| \quad \text{exists.}$$

The limit is denoted by $\|x + \sum_{i=1}^k a_i e_i\|$ and defines a norm on $X \oplus E$ where $E = [e_i]$. E is called a **spreading model** of X and $X \oplus E$ is called a **spreading model** of (x_i) over X. (e_i) is **1-unconditional over** X if for all $x \in X$, $(\alpha_i)_1^k \subseteq \mathbb{R}$ and (ε_i) with $|\varepsilon_i| = 1$ for all i,

$$\Big\| x + \sum_{i=1}^k \alpha_i e_i \Big\| = \Big\| x + \sum_{i=1}^k \varepsilon_i \alpha_i e_i \Big\| .$$

COROLLARY 7: *Every large sequence (F_n) of finite dimensional subspaces of an infinite dimensional Banach space X has a large refinement (G_n) with the following property: All sequences (x_n), with $x_n \in S_{G_n}$ for $n \in \mathbb{N}$, have the same spreading model $E = [e_i]$ over X. In particular (e_i) is 1-unconditional over X. Moreover, G_n can be chosen, so that for all $\varepsilon > 0$, $k \in \mathbb{N}$ and $x \in X$ there exists $k_0 \in \mathbb{N}$ such that if $k_0 \leq n_1 < n_2 < \cdots < n_k$, then*

$$\left| \left\| x + \sum_{i=1}^{k} \alpha_i e_i \right\| - \left\| x + \sum_{i=1}^{k} \alpha_i x_i \right\| \right| < \varepsilon$$

whenever $x_i \in S_{G_{n_i}}$, $i = 1, \ldots, k$, and $(\alpha_i)_{i=1}^k \in \mathrm{Ba}(\ell_\infty^k)$.

As usual, there is a corresponding "block refinement" version. Corollary 7 follows from Theorem 6 and a standard diagonal argument using the Lipschitz functions $f_x(y) = \|x + y\|$ as x ranges over a dense subset of X. The result that every Banach space X has a spreading model which is 1-unconditional over X is due to the second named author, see [R4], [R5].

We note finally an application of Theorems 1 and 6 to the Banach-Saks property. The following principle was discovered in 1975 (cf. [R2]; a proof may be found in [BL]).

Given (x_j) a semi-normalized weakly null sequence in a Banach space, there is a subsequence (x'_j) so that either (x'_j) has a spreading model isomorphic to ℓ_1, or $\frac{1}{n}\|\sum_{j=1}^n x''_j\| \to 0$ as $n \to \infty$ for all further subsequences (x''_j) of (x'_j).

Now in fact one may assume in any case that (x'_j) generates a spreading model, with basis (b_j) say; then the second alternative occurs precisely when (b_j) itself is weakly null. In this case, one has $\|\frac{1}{n}\sum_{j=1}^n b_j\| \to 0$ as $n \to \infty$. Then, e.g., setting $\varepsilon_n = \frac{2}{n}\|\sum_{j=1}^n b_j\|$, (x'_j) can be chosen so that $\frac{1}{n}\|\sum_{j=1}^n x''_j\| \leq \varepsilon_n$ for all subsequences (x''_j) of (x'_j).

The following result now follows immediately from Theorem 1 and Corollary 7.

COROLLARY 8: *Let (F_j) be a large sequence of finite dimensional subspaces of a Banach space X, so that no normalized sequence (f_j), with $f_j \in F_j$ for all j, has a subsequence equivalent to the ℓ_1-basis. Then there is a large weakly null* FDD *refinement (G_j) of (F_j), having one of the following mutually exclusive alternatives:*

(1) (G_j) *is uniformly anti-Banach-Saks; that is, there is a* $\delta > 0$ *so that*

$$\varliminf_{n\to\infty} \frac{1}{n} \left\| \sum_{j=1}^{n} g_j \right\| \geq \delta \varliminf_{n\to\infty} \|g_n\|$$

for all strictly increasing sequences (n_j) *in* $\mathbb{N}$ *and all sequences* $(g_j) \in \prod_{j=1}^{\infty} \mathrm{Ba}\, G_{n_j}$.

(2) (G_j) *is uniformly Banach-Saks; that is, there is a sequence* (ε_j) *of positive numbers tending to zero so that*

$$\frac{1}{n} \left\| \sum_{j=1}^{n} g_j \right\| \leq \varepsilon_n \quad \text{for all } n ,$$

all strictly increasing sequences (n_j) *in* $\mathbb{N}$, *and all sequences* $(g_j) \in \prod_{j=1}^{\infty} \mathrm{Ba}\ G_{n_j}$. *Moreover if the* F_n*'s have bases* $\mathbf{b}_n$ *with uniformly bounded basis constants,* (G_n) *may be chosen to be a block refinement of* (F_n) *with respect to* $(\mathbf{b}_n)$.

2. Proofs of Theorems 1 and 6

Proof of Theorem 1: Without loss of generality we can assume that $X = C(K)$, the space of all real or complex valued continuous functions on a compact metric space K. For $f \in C(K)$ we let $f^+ = \max(f, 0)$ in the real case; in the complex case we put $f^+ = \min((\mathrm{Re}\, f)^+, (\mathrm{Im}\, f)^+)$. For $A \subset K$ we let $\|\cdot\|_A$ be the seminorm on $C(K)$ defined by $\|f\|_A = \sup_{\xi\in A} |f(\xi)|$. Let (F_n) be a large sequence of finite dimensional subspaces of $C(K)$. Since (F_n) *has* a large FDD-refinement, we assume without loss of generality that (F_n) is already an FDD.

We consider the following two cases.

CASE 1:

(1) For all nonempty closed sets $\tilde{K} \subset K$, all $\varepsilon > 0$ and all large refinements (H_n) of (F_n) there is a relatively open set $U \subset \tilde{K}$, $U \neq \emptyset$, and a large refinement $(\tilde{H}_n)$ of (H_n) so that

$$\sup \|f\|_U < \varepsilon\ , \quad \text{for } f \in \bigcup_{n\in\mathbb{N}} S_{\tilde{H}_n} .$$

CASE 2:

(2) There are a nonempty closed set $K_0 \subset K$, $\varepsilon_0 > 0$ and a large refinement (H_n) of (F_n) so that for all nonempty and relatively open sets $U \subset K_0$ and all further large refinements $(\tilde{H}_n)$ of (H_n),

$$\liminf_{n\to\infty} \sup_{h\in S_{\tilde{H}_n}} \|h\|_U > \varepsilon_0 .$$

Clearly, cases 1 and 2 are mutually exclusive and the failure of one implies the other holds. We will show that assuming case 1, we can find a weakly null large refinement (G_n) of (F_n). Assuming case 2, we shall produce a uniformly-ℓ_1 large refinement (G_n) of (F_n).

Assume that (1) is satisfied and let $\varepsilon > 0$ be arbitrary. Let $K^{(0)} = K$ and $(H_n^{(0)}) = (F_n)$. We will choose by transfinite induction for each $\alpha < \omega_1$ (where ω_1 is the first uncountable ordinal), a closed subset $K^{(\alpha)}$ of K and a large refinement $(H_n^{(\alpha)})$ of (F_n), so that

(3) $K^{(\beta)} \subseteq K^{(\alpha)}$ and, if $K^{(\alpha)} \neq \emptyset$, then $K^{(\beta)} \subsetneq K^{(\alpha)}$, whenever $\alpha < \beta$.

(4) Except for perhaps finitely many elements, $(H_n^{(\beta)})$ is a refinement of $(H_n^{(\alpha)})$ whenever $\alpha < \beta$.

(5) For all $\xi \in K \setminus K^{(\alpha)}$,

$$\limsup_{n\to\infty} \sup_{f\in S_{H_n^{(\alpha)}}} |f(\xi)| \le \varepsilon$$

Assume that for some $\alpha < \omega_1$, $(K^{(\gamma)})_{\gamma<\alpha}$ and $(H_n^{(\gamma)})_{\gamma<\alpha}$ have been chosen. If $\alpha = \gamma + 1$ and $K^{(\gamma)} = \emptyset$ set $K^{(\alpha)} = \emptyset$ and $(H_n^{(\alpha)}) = (H_n^{(\gamma)})$. If $\alpha = \gamma + 1$ and $K^{(\gamma)} \neq \emptyset$, by (1) there exists a large refinement $(H_n^{(\alpha)})$ of $(H_n^{(\gamma)})$ and a relatively open set $U \subset K^{(\gamma)}$, $U \neq \emptyset$, so that

$$\|f\|_U < \varepsilon \text{ for all } f \in \bigcup_{n\in\mathbb{N}} S_{H_n^{(\gamma)}} .$$

Set $K^{(\alpha)} = K^{(\gamma)} \setminus U$.

If $\alpha = \lim_{n\to\infty} \gamma_n$ for some strictly increasing sequence (γ_n), set $K_\alpha = \bigcap_{n\in\mathbb{N}} K_{\gamma_n}$ and let $(H_n^{(\alpha)})$ be a "diagonal sequence" of $(H_n^{(\gamma_m)})_{n,m\in\mathbb{N}}$, chosen such that for each m, except for perhaps finitely many terms, $(H_n^{(\alpha)})$ is a large refinement of $(H_n^{(\gamma_m)})$.

Since K is compact and metric, (thus K satisfies the Lindelöff condition) we conclude that for some $\alpha < \omega_1$, $K^{(\beta)} = K^{(\alpha)}$ for $\alpha \le \beta < \omega_1$. By (3) it follows that $K^{(\alpha)} = \emptyset$ and from (5) it follows that for all $\xi \in K$,

$$\limsup_{n\to\infty} \sup_{f\in S_{H_n^{(\alpha)}}} |f(\xi)| \le \varepsilon .$$

We let $(H_n^{(\varepsilon)}) := (H_n^{(\alpha)})$. Repeating this argument for a sequence $(\varepsilon_m) \subset \mathbb{R}_+$ with $\varepsilon_m \downarrow 0$ one obtains for each $m \in \mathbb{N}$, a large refinement $(H_n^{(\varepsilon_m)})_{n\in\mathbb{N}}$, of $(H_n^{(\varepsilon_{m-1})})$, satisfying for all $\xi \in K$,

$$\limsup_{n\to\infty} \sup_{f\in S_{H_n^{(\varepsilon_m)}}} |f(\xi)| \le \varepsilon_m .$$

If we let (G_n) be a diagonal sequence of $(H_n^{(\varepsilon_m)})_{n,m\in\mathbb{N}}$, still satisfying $\lim_{n\to\infty} \dim(G_n) = \infty$, we deduce that for all $\xi \in K$,

$$\lim_{n\to\infty} \sup_{f\in S_{G_n}} |f(\xi)| = 0 .$$

Thus (G_n) is a weakly null large refinement of (F_n).

We now assume that (2) is satisfied and let $K_0 \subset K$, $\varepsilon_0 > 0$ and (H_n) be as in (2). Let $\varepsilon_1 = \varepsilon_0$ in the real case and $\varepsilon_1 = \varepsilon_0/\sqrt{2}$ in the complex case. Let D be a countable dense subset of K_0. By passing to a large refinement of (H_n) we can assume that

$$\lim_{n\to\infty} \sup_{f\in S_{H_n}} |f(\xi)| = 0 \ \text{ for all } \ \xi \in D . \tag{6}$$

Indeed, let $d_1, d_2, \dots$ be an enumeration of D and let $m_1 < m_2 < \cdots$ be such that $\dim H_{m_n} \ge 2n$; then set $H'_n = \{x \in H_{m_n} : x(d_i) = 0 \text{ for } 1 \le i \le n\}$. Now $\dim H'_n \ge n$ for all n, so (H'_n) is the desired large refinement. Let $\varepsilon_1/34 > \delta > 0$. By induction we will choose an increasing sequence of integers (k_n) and for each n, a subspace G_n of H_{k_n} and a finite set Π_n consisting of nonempty relatively open subsets of K_0 so that the following conditions are satisfied:

(7) $\dim(G_n) \ge n$,

and

(8) For every $g \in S_{G_n}$, and every $U \in \Pi_{n-1}$ (let $\Pi_0 = \{K_0\}$) there are $U_1, U_2 \in \Pi_n$, $U_1 \cup U_2 \subseteq U$, so that

$$g^+|_{U_1} \ge \varepsilon_1 - \delta \quad \text{and} \quad \|g\|_{U_2} \le \delta .$$

Once we have chosen (G_n) in this way we conclude that (G_n) must be uniformly-ℓ_1. To see this, fix (f_n) with $f_n \in S_{G_n}$ for all $n \in \mathbb{N}$. For each n, let $A_n = \{k \in K\colon f_n(k) > \varepsilon_1 - \delta\}$ and $B_n = \{k \in K\colon |f_n(k)| < \delta\}$. Evidently $A_n \cap B_n = \emptyset$ for all n. We shall show that (A_n, B_n) is an independent sequence of pairs, in the terminology of [R1]. Once this is done, a refinement of the argument in [R1] yields that (f_n) is $\frac{16}{\varepsilon_1}$-equivalent to the ℓ_1-basis.

Indeed, we first can inductively choose sets $(U_i^{(n)}\colon i = 1, 2, \ldots, 2^n) \subset \Pi^{(n)}$ so that

$$f_n^+\big|_{\bigcup_{i=1}^{2^{n-1}} U_{2i-1}^{(n)}} > \varepsilon_1 - \delta \quad \text{and} \quad \|f_n\|_{\bigcup_{i=1}^{2^{n-1}} U_{2i}^{(n)}} < \delta$$

and so that $U_{2j}^{(n)} \cup U_{2j-1}^{(n)} \subset U_j^{(n-1)}$ for $n \in \mathbb{N}$ and $j = 1, 2, \ldots, 2^{n-1}$. Now fix N, I and J non-empty disjoint subsets of $\{1, \ldots, N\}$, say with $I \cup J = \{1, \ldots, N\}$. We see that $\bigcap_{n\in I} A_n \cap \bigcap_{n\in J} B_n$ is non-empty by defining the following sequence of sets $C_0, C_1, \ldots, C_N$: Let $U_1^0 = K_0 = C_0$, $1 \le n \le N$, and suppose C_{n-1} is chosen with $C_{n-1} = U_j^{(n-1)}$ for some $1 \le j \le 2^{n-1}$. If $n \in I$, set $C_n = U_{2j-1}^{(n)}$, otherwise set $C_n = U_{2j}^{(n)}$. Then the C_n's satisfy that $\bigcap_{n=1}^N C_n \neq \emptyset$ and for all n, $C_n \subset A_n$ if $n \in I$, $C_n \subset B_n$ if $n \in J$.

Now let $\sum_{j=1}^N |a_j| = 1$ with $a_j = b_j + ic_j$ for $j \le N$. By multiplying by -1, i or $-i$ if necessary we may assume that $\sum_{j=1}^N b_j^+ \ge 1/4$. Let $I = \{j \le N\colon b_j \ge 0$ and $c_j \ge 0\}$ and $J = \{j \le N\colon b_j \ge 0$ and $c_j < 0\}$. Thus either

$$\sum_{j\in I}(b_j + c_j) \ge \frac{1}{8} \qquad \text{or} \qquad \sum_{j\in J}(b_j - c_j) \ge \frac{1}{8}\,.$$

Suppose the first sum exceeds $1/8$. Now by the independence of (A_n, B_n), choose $k \in K$ such that $f_j^+(k) > \varepsilon_1 - \delta$ for $j \in I$ and $|f_j(k)| < \delta$ for $j \notin I$. Let $f_j(k) = B_j + iC_j$. Then

$$\begin{aligned}\Big|\sum_{j=1}^N a_j f_j(k)\Big| &\ge \Big|\mathrm{Im}\Big(\sum_{j=1}^N a_j f_j(k)\Big)\Big| \\ &= \Big|\sum_{j=1}^n (b_j C_j + B_j c_j)\Big| \\ &\ge \sum_{j\in I}(b_j C_j + B_j c_j) - \sum_{j\notin I}|b_j C_j + B_j c_j| \\ &\ge \frac{(\varepsilon_1 - \delta)}{8} - 2\delta > \frac{\varepsilon_1}{16}\,.\end{aligned}$$

A similar estimate ensues if the second sum exceeds 1/8. Thus (f_n) is indeed $\frac{16}{\varepsilon_1}$-equivalent to the ℓ^1-basis.

Assume that for some $n \geq 1$, Π_{n-1} and k_{n-1} (let $k_0 = 0$) are chosen. Now consider the finite family of Lipschitz functions defined on $C(K)$ by $f \mapsto \|f^+\|_U$, $U \in \Pi_{n-1}$. Since (H_n) is large, we may use the first stabilization principle in order to pass to a large refinement $(\tilde{H}_i)$ of $(H_i)_{i>k_{n-1}}$ so that for some family $(a_i^{(U)}: U \in \Pi_{n-1}, i \in \mathbb{N})$ in $\mathbb{R}^+$ we have

$$a_i^{(U)} - \frac{\delta}{4} < \|f^+\|_U < a_i^{(U)} + \frac{\delta}{4}$$

whenever $U \in \Pi_{n-1}$, $i \in \mathbb{N}$ and $f \in S_{\tilde{H}_i}$. From (2) we deduce that there exists $i_0 \in \mathbb{N}$ so that for all $i \geq i_0$ and $U \in \Pi_{n-1}$ we have $a_i^{(U)} \geq \varepsilon_1 - \frac{\delta}{4}$. Indeed, in the real case we only have to observe that if $\|f\|_U \geq \varepsilon_0$ then $\|f^+\|_U \geq \varepsilon_0$ or $\|(-f)^+\|_U \geq \varepsilon_0$; in the complex we find for any $f \in C(K)$ for which $\|f\|_U > \varepsilon_0$, a point $\xi \in U$ with $|f(\xi)| > \varepsilon_0$ and then a complex number a, with $|a| = 1$, so that $\mathrm{Re}(a \cdot f(\xi)) = \mathrm{Im}(a \cdot f(\xi)) = (a \cdot f(\xi))^+$. Thus $\|(a \cdot f)^+\|_U \geq \frac{1}{\sqrt{2}}\|f\|_U > \varepsilon_1$. We deduce that

$$\|f^+\|_U > \varepsilon_1 - \frac{\delta}{2} \tag{9}$$

for all $U \in \Pi_{n-1}$, $i \geq i_0$ and $f \in S_{\tilde{H}_i}$.

Now using (6), we pick, for each $U \in \Pi_{n-1}$, an element $\xi_U \in U \cap D$ and find an $i_1 \geq i_0$ so that $\dim(\tilde{H}_{i_1}) \geq n$ and so that

$$\sup_{f \in S_{\tilde{H}_{i_1}}} |f(\xi_U)| < \frac{\delta}{2}\,. \tag{10}$$

Let $(f_s)_{s=1}^{\ell}$ be a finite $\frac{\delta}{2}$-net for $S_{\tilde{H}_{i_1}}$. We find by (9) and (10) for each $U \in \Pi_{n-1}$, non-empty open subsets $V_0^{(U)}, V_1^{(U)}, \ldots, V_\ell^{(U)}$ so that $f_s^+|_{V_s^{(U)}} > \varepsilon_1 - \frac{\delta}{2}$ and $\|f_s\|_{V_0^{(U)}} < \frac{\delta}{2}$, for $s = 1, 2, \ldots, \ell$. This implies that for all $f \in S_{\tilde{H}_{i_1}}$ we have $\|f\|_{V_0^{(U)}} < \delta$, and for some $1 \leq s \leq \ell$ (namely the s for which $\|f - f_s\| < \frac{\delta}{2}$) we have $f^+|_{V_s^{(U)}} > \varepsilon_1 - \delta$. Set

$$\Pi_n = \left\{V_0^{(U)}: U \in \Pi_{n-1}\right\} \cup \left\{V_s^{(U)}: 1 \leq s \leq \ell\,,\ U \in \Pi_{n-1}\right\}\,,$$

$G_n = \tilde{H}_{i_1}$, and choose $k_n > k_{n-1}$ so that $\tilde{H}_{i_1} \subset H_{k_n}$. This completes the induction and thus the proof of the first version of Theorem 1.

The "block-version" of Theorem 1 is proved in exactly the same way using the second stabilization principle instead of the first. We need only note that block refinements could be taken wherever we took simple refinements. ∎

Proof of Theorem 6: As in the proof of Theorem 1 we will only show the first version of Theorem 6. The "block-version" is left to the reader. We shall assume that X is a Banach space over $\mathbb{R}$. The complex case does not provide any further difficulties.

Let f: $X \to \mathbb{R}$ be Lipschitz and let $\varepsilon_n \downarrow 0$. We accomplish the proof by induction, insuring the conditions in the Theorem for a fixed $k \geq 2$. Precisely, we shall choose for each k, a large sequence $(G_n^{(k)})$ of finite dimensional subspaces so that $(G_n^{(k+1)})$ is a refinement of $(G_n^{(k)})$ $((G_n^{(1)}) \equiv (F_n))$, and a function $C^{(k)}$: $\mathrm{Ba}(\ell_\infty^k) \to \mathbb{R}$, so that

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) \overset{\varepsilon_{n_1}}{=} C^{(k)}(\alpha_1, \ldots, \alpha_k)$$

whenever $(\alpha_1, \ldots, \alpha_k) \in \mathrm{Ba}(\ell_\infty^k)$ and $x_k \in S(G_{n_i}^{(k)})$, for all $1 \leq n_1 < n_2 < \cdots < n_k$.

Once this is done, then by diagonalization we finally find a large refinement (G_n) of (F_n) and functions $C^{(k)}$: $\mathrm{Ba}(\ell_\infty^k) \to \mathbb{R}$, $k \in \mathbb{N}$ so that for all $k \in \mathbb{N}$ and all $k \leq n_1 < n_2 < \cdots < n_k$ we have

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k) \overset{\varepsilon_{n_1}}{=} C^{(k)}(\alpha_1, \ldots, \alpha_k)$$

whenever $x_i \in S_{G_{n_i}}$, for $i = 1, 2, \ldots, k$.

Clearly we have that

$$C^{(k)}(\alpha_1, \alpha_2, \ldots, \alpha_k) = C^{(k+s)}(\alpha_1, \alpha_2, \ldots, \alpha_k, 0, 0, \ldots, 0)$$

for $k, s \in \mathbb{N}$ and $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathrm{Ba}(\ell_\infty^k)$, and, thus, if we put

$$\tilde{f}(\alpha_1, \ldots, \alpha_k, 0, 0, \ldots) = C^{(k)}(\alpha_1, \alpha_2, \ldots, \alpha_k)\ ,$$

for $k \in \mathbb{N}$ and $(\alpha_i)_{i=1}^k \in \mathrm{Ba}(\ell_\infty^k)$, $\tilde{f}$ has the required properties.

We now indicate in detail how to carry this out for $k = 2$. First note the following

FACT: *Let $g\colon S_X \to \mathbb{R}$ be Lipschitz and let (L_n) be any large sequence of finite dimensional subspaces of X. Let $\delta_n \downarrow 0$. There exist a large refinement $(\tilde{L}_n)$ of (L_n) and $C \in \mathbb{R}$ such that for all n and $y \in S_{\tilde{L}_n}$, we have $g(y) \overset{\delta_n}{=} C$.*

This follows easily from the first stabilization theorem. One first obtains a large refinement $(\tilde{\tilde{L}}_n)$ of (L_n) and $(C_n) \subseteq \mathbb{R}$ such that $g(y) \overset{\delta_n/2}{=} C_n$ for $y \in S_{\tilde{\tilde{L}}_n}$. (C_n) is bounded so for some subsequence (C_{k_n}) and $C \in \mathbb{R}$, $|C_{k_n} - C| < \delta_n/2$ for all n. Let $\tilde{L}_n = \tilde{\tilde{L}}_{k_n}$.

Let $H_1 \equiv F_1$. Choose finite sets $D_1 \subseteq D_2 \subseteq \cdots \subseteq \mathrm{Ba}(\ell^2_\infty)$ and $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \cdots \subseteq S_{H_1}$ so that for all n, D_n is an ε_n-net for $\mathrm{Ba}(\ell^2_\infty)$ and $\mathcal{D}_n$ is an ε_n-net for S_{H_1}. For $x \in S_{H_1}$ and $(\alpha,\beta) \in \mathrm{Ba}(\ell^2_\infty)$, $y \mapsto f(\alpha x + \beta y)$ is a Lipschitz function on X. Thus by iterating the fact above a finite number of times we obtain a large refinement $(F_n^{(1,1)})_{n=1}^\infty$ of (F_n) and $(C(\alpha,\beta,x))_{(\alpha,\beta,x)\in D_1\times\mathcal{D}_1} \subseteq \mathbb{R}$ such that for all $(\alpha,\beta) \in D_1$, $x \in \mathcal{D}_1$ and $y \in F_n^{(1,1)}$,

$$f(\alpha x + \beta y) \overset{\varepsilon_n}{=} C(\alpha,\beta,x) .$$

Repeating this argument inductively we obtain for all $k \in \mathbb{N}$, a large refinement $(F_n^{(1,k)})_{n=1}^\infty$ of $(F_n^{(1,k-1)})_{n=1}^\infty$ and $(C(\alpha,\beta,x))_{(\alpha,\beta,x)\in D_k\times\mathcal{D}_k}$ such that

$$f(\alpha x + \beta y) \overset{\varepsilon_n}{=} C(\alpha,\beta,x)$$

if $(\alpha,\beta) \in D_k$, $x \in \mathcal{D}_k$ and $y \in F_n^{(1,k)}$. By diagonalization we obtain a large refinement $(F_n^{(1)})_{n=1}^\infty$ of (F_n) with the property

(i) For $k \in \mathbb{N}$, $(\alpha,\beta,x) \in D_k \times \mathcal{D}_k$, $n \ge k$, and $y \in F_n^{(1)}$,

$$f(\alpha x + \beta y) \overset{\varepsilon_n}{=} C(\alpha,\beta,x) .$$

Suppose that the Lipschitz constant of f is $K \ge 1$, i.e., $|f(x) - f(y)| \le K\|x - y\|$. Then for $(\alpha,\beta),(\alpha',\beta') \in \mathrm{Ba}(\ell^2_\infty)$, $x,x' \in S_{H_1}$, and $\|y\| = 1$, we have

$$\begin{aligned}|f(\alpha x + \beta y) - f(\alpha' x' + \beta' y)| &\le K\|(\alpha x - \alpha' x) + (\alpha' x - \alpha' x') + (\beta - \beta')y\| \\ &\le K\big(|\alpha - \alpha'| + |\beta - \beta'| + \|x - x'\|\big) .\end{aligned}$$

From this and (i) we obtain

(ii) $\quad |C(\alpha,\beta,x) - C(\alpha',\beta',x')| \le K\big[|\alpha - \alpha'| + |\beta - \beta'| + \|x - x'\|\big]$

whenever $(\alpha,\beta),(\alpha',\beta') \in \bigcup D_n$ and $x,x' \in \bigcup \mathcal{D}_n$. Thus we can uniquely extend $C(\alpha,\beta,x)$ to a function $C^{[1]}\colon \mathrm{Ba}(\ell^2_\infty) \times S_{H_1} \to \mathbb{R}$ which satisfies (ii) for all

$(\alpha,\beta),(\alpha',\beta') \in \mathrm{Ba}(\ell_\infty^2)$ and $x,x' \in S_{H_1}$. Furthermore, by replacing $(F_n^{(1)})$ by an appropriate subsequence, we may assume that

(iii) For $n \in \mathbb{N}$, $(\alpha,\beta,x) \in \mathrm{Ba}(\ell_\infty^2) \times S_{H_1}$ and $y \in S_{F_n^{(1)}}$,

$$f(\alpha x + \beta y) \overset{\varepsilon_n}{=} C^{[1]}(\alpha,\beta,x) .$$

Set $H_2 = F_{n_2}^{(1)}$ where n_2 is chosen so that $\dim H_2 > \dim H_1$. Proceeding as above we obtain a function

$$C^{[2]}\colon \mathrm{Ba}(\ell_\infty^2) \times S_{H_2} \to \mathbb{R}$$

and a large refinement $(F_n^{(2)})$ of $(F_n^{(1)})$ so that

(iv) $|C^{[2]}(\alpha,\beta,x) - C^{[2]}(\alpha',\beta',x')| \le K\big[|\alpha-\alpha'| + |\beta-\beta'| + \|x-x'\|\big]$

for all $(\alpha,\beta),(\alpha',\beta') \in \mathrm{Ba}(\ell_\infty^2)$ and $x,x' \in S_{H_2}$ and

(v) $f(\alpha x + \beta y) \overset{\varepsilon_n}{=} C^{[2]}(\alpha,\beta,x)$

for all $x \in S_{H_2}$, $(\alpha,\beta) \in \mathrm{Ba}(\ell_\infty^2)$ and $y \in S_{F_n^{(2)}}$.

We continue in this manner obtaining a large refinement (H_n) of (F_n) and, for $k \in \mathbb{N}$, functions $C^{[k]}\colon \mathrm{Ba}(\ell_\infty^2) \times S_{H_k} \to \mathbb{R}$ satisfying

(vi) $|C^{[k]}(\alpha,\beta,x) - C^{[k]}(\alpha',\beta',x')| \le K\big[|\alpha-\alpha'| + |\beta-\beta'| + \|x-x'\|\big]$

for $(\alpha,\beta),(\alpha',\beta') \in \mathrm{Ba}(\ell_\infty^2)$ and $x,x' \in S_{H_k}$ and

(vii) $f(\alpha x + \beta y) \overset{\varepsilon_n}{=} C^{[k]}(\alpha,\beta,x)$

for all $(\alpha,\beta) \in \mathrm{Ba}(\ell_\infty^2)$, $x \in S_{H_k}$ and $y \in S_{H_n}$ with $n > k$.

(Actually it might be necessary to pass to a subsequence of (H_n) to obtain the precise estimate (vii).)

We now apply the first stabilization result to finite sets of functions $C^{[k]}(\alpha,\beta,\cdot)$. Let $n \in \mathbb{N}$ and $0 \le \bar\varepsilon \le \min\{|\alpha-\alpha'| + |\beta-\beta'|\colon (\alpha,\beta) \ne (\alpha',\beta') \in D_n\}$. Consider for a fixed k the Lipschitz functions $C^{[k]}(\alpha,\beta,\cdot)\colon S_{H_k} \to \mathbb{R}$ for $(\alpha,\beta) \in D_n$. If $\dim H_k$ is sufficiently large there exists $\tilde H_k \subseteq H_k$, $\dim \tilde H_k \ge n$ and $(C^k(\alpha,\beta))_{(\alpha,\beta)\in D_n} \subseteq \mathbb{R}$ so that $C^{[k]}(\alpha,\beta,x) \overset{\bar\varepsilon}{=} C^k(\alpha,\beta)$ for all $x \in S_{\tilde H_k}$ and $(\alpha,\beta) \in D_n$. Thus this plus (vi) yields

$$\begin{aligned}|C^k(\alpha,\beta) - C^k(\alpha',\beta')| &\le K\big[|\alpha-\alpha'| + |\beta-\beta'|\big] + 2\bar\varepsilon \\ &\le 3K\big[|\alpha-\alpha'| + |\beta-\beta'|\big]\end{aligned}$$

for $(\alpha,\beta),(\alpha',\beta') \in D_j$. The last inequality holds by the choice of $\bar\varepsilon$ and the fact that $K \ge 1$.

We inductively use this argument for the parameters $(n, \bar{\varepsilon}_n)$ where $\bar{\varepsilon}_n \downarrow 0$ rapidly chosen depending on (D_n) and (ε_n). We obtain a large refinement (I_n) of (H_n) with $\dim I_n \geq n$ and functions $C^n\colon D_n \to \mathbb{R}$ satisfying

(viii) $|C^n(\alpha,\beta) - C^n(\alpha',\beta')| \leq 3K\big[|\alpha-\alpha'| + |\beta-\beta'|\big]$ for $(\alpha,\beta),(\alpha',\beta') \in D_n$

and

(ix) For all $x \in S_{I_n}$ and $(\alpha,\beta) \in D_n$,

$$f(\alpha x + \beta y) \overset{\varepsilon_n}{=} C^n(\alpha,\beta) \text{ whenever } y \in S_{I_q}\,, \qquad q > n\,.$$

For $n \in \mathbb{N}$, the function $(C^k|_{D_n})_{k \geq n}$ are uniformly Lipschitz. Thus by a compactness argument we can find a Lipschitz function $C^{(2)}\colon \cup D_j \to \mathbb{R}$ and $k_1 < k_2 < \cdots$ so that for all n and $(\alpha,\beta) \in D_n$,

$$C^{(2)}(\alpha,\beta) \overset{\varepsilon_n}{=} C^{k_n}(\alpha,\beta)\,.$$

$C^{(2)}$ thus uniquely extends to a continuous function $C^{(2)}\colon \mathrm{Ba}(\ell_\infty^2) \to \mathbb{R}$. Letting $(G_n^{(2)})_{n=1}^\infty$ be a suitable subsequence of (I_{k_n}) we obtain

(x) $f(\alpha x_{n_1} + \beta x_{n_2}) \overset{\varepsilon_{n_1}}{=} C^{(2)}(\alpha,\beta)$ for all $(\alpha,\beta) \in \mathrm{Ba}(\ell_\infty^2)$, $n_1 < n_2$, $x_{n_1} \in S_{G_{n_1}^{(2)}}$ and $x_{n_2} \in S_{G_{n_2}^{(2)}}$

which was what was needed to be proved in the case $k = 2$. ∎

3. A Sketch of the Proof of the Second Stabilization Principle

The reader unfamiliar with Lemberg's proof might first wish to read that argument (see [MS, Ch.12]). In order to shorten the proof we will not only use Lemberg's proof of Krivine's theorem but also the quantitative version of this theorem.

THEOREM 9 (see [R3]): *For every $C > 1$, $\varepsilon > 0$ and $k \in \mathbb{N}$, there is an $n = n(C,\varepsilon,k) \in \mathbb{N}$ so that: If F is a Banach space of dimension n and if $(f_i)_{i=1}^n$ is a basis of F having basis constant not exceeding C, then there exists a block basis $(g_i)_{i=1}^k$ of $(f_i)_{i=1}^n$ and a $p \in [1,\infty]$ so that $(g_i)_{i=1}^k$ is $(1+\varepsilon)$-equivalent to the unit basis of ℓ_p^k.*

In view of Theorem 9 we only have to prove the second stabilization principle for finite dimensional ℓ_p-spaces. Using a compactness argument, similar to the argument of [R3] by which Theorem 9 was deduced from the finite dimensional version of Krivine's theorem, we only have to show the following claim.

CLAIM 1: *Let $X=\ell_p$, $1\le p<\infty$, or $X=c_0$, and let $f\colon X\to\mathbb{R}$ be a Lipschitz function. For each $\varepsilon>0$ and $k\in\mathbb{N}$ there exists a block basis $(y_i)_{i=1}^k$ of (e_i) (the unit vector basis of X) so that* $\operatorname{osc}(f|S_{[y_i]_{i=1}^k})<\varepsilon$.

Proof of Claim 1: We need some notation. For $x,y\in c_{00}$ we say x and y have the **same distribution**, and write $x\stackrel{\text{dist}}{=}y$, if $x=\sum_{i=1}^k\alpha_ie_{n_i}$ and $y=\sum_{i=1}^k\alpha_ie_{m_i}$ for some $k\in\mathbb{N}$, $(\alpha_i)_{i=1}^k\subset\mathbb{R}$, resp. $\mathbb{C}$, and $n_1<n_2<\cdots n_k$ and $m_1<m_2<\cdots m_k$. We define for $x,y\in X\cap c_{00}$,

$$\operatorname{dis}(x,y)=\inf\left\{\|\bar{x}-\bar{y}\|\colon \bar{x}\stackrel{\text{dist}}{=}x \ \text{ and } \ \bar{y}\stackrel{\text{dist}}{=}y\right\}\ .$$

For $x\in c_{00}$, we let $\operatorname{supp}(x)=\{i\in\mathbb{N}\colon x_i\neq 0\}$, and write $x<y$ for $x,y\in c_{00}$ if $\max(\operatorname{supp}(x))<\min(\operatorname{supp}(y))$.

We first reduce claim 1 to the case that f is 1-unconditional and 1-spreading. By this we mean that $f(\sum\alpha_ie_i)=f(\sum|\alpha_i|e_{n_i})$ for all $\sum\alpha_ie_i\in X$ and all strictly increasing sequences $(n_i)\subset\mathbb{N}$.

In order to reduce claim 1 to the 1-unconditional and 1-spreading case we first pass to a sequence $n_i\subset\mathbb{N}$ for which

$$f^{(1)}\left(\sum_{i=1}^{\ell}\alpha_ie_i\right)=\lim_{k_1\to\infty}\ \lim_{k_2\to\infty}\cdots\lim_{k_\ell\to\infty}f\left(\sum_{i=1}^{\ell}\alpha_ie_{n_{k_i}}\right)$$

exists for all ℓ and scalars $\alpha_1,\alpha_2,\dots,\alpha_\ell$. It follows that $f^{(1)}$ is 1-spreading on X. If X is defined over $\mathbb{R}$ we let $\ell_n=2$, for $n\in\mathbb{N}$, and put $(\xi_i)_{i=1}^{\ell_n}=(1,-1)$. If X is defined over $\mathbb{C}$ we let $\ell_n=n$, for $n\in\mathbb{N}$, and $(\xi_j)_{j=1}^{\ell_n}=e^{(i2\pi j/n)}$. If $X=\ell_p$, $1\le p<\infty$, let (u_n) be a sequence in X with $u_1<u_2<\cdots$ and

$$u_n\stackrel{\text{dist}}{=}\frac{1}{(n\ell_n)^{1/p}}\sum_{s=1}^{n}\ \sum_{t=1}^{\ell_n}\xi_te_{(s-1)\ell_n+t}\ ,\quad\text{for}\ \ n\in\mathbb{N}\ .$$

If $X=c_0$ we let (u_n) be a sequence in X, with $u_1<u_2<\cdots$, and

$$u_n\stackrel{\text{dist}}{=}\sum_{s=1}^{n}\frac{s}{n}\cdot\sum_{t=1}^{\ell_n}\xi_te_{(s-1)\ell_n+t}+\sum_{s=1}^{n-1}\frac{n-s}{n}\sum_{t=1}^{\ell_n}\xi_te_{(n+s-1)\ell_n+t}$$

Note that (u_n) is normalized, and that from the fact that $f^{(1)}$ is 1-spreading it follows that for some sequence $\varepsilon_n\downarrow 0$ and some subsequence $(\tilde{u}_n)$ of (u_n),

$$f^{(1)}\left(\sum_{j=1}^{k}\alpha_j\tilde{u}_{j+n}\right)\stackrel{\varepsilon_n}{=}f^{(1)}\left(\sum_{j=1}^{k}\sigma_j\alpha_j\tilde{u}_{j+n}\right)$$

whenever $k, n \in \mathbb{N}$, $|\sigma_j| = 1$, for $j = 1, 2, \ldots, k$, and $\|\sum_{j=1}^{k} \alpha_j e_j\| = 1$.

Pass now to a subsequence $(n_i) \subset \mathbb{N}$ for which

$$f^{(2)}\left(\sum_{i=1}^{\ell} \alpha_i e_i\right) \equiv \lim_{k_1\to\infty} \lim_{k_2\to\infty} \ldots \lim_{k_\ell\to\infty} f^{(1)}\left(\sum_{i=1}^{\ell} \alpha_i \tilde{u}_{n_{k_i}}\right)$$

exists, whenever $\ell \in \mathbb{N}$ and $(\alpha_i)_{i=1}^{\ell} \in c_{00}$. $f^{(2)}$ is 1-unconditional and 1-spreading and we need only prove that claim 1 is true for $f^{(2)}$. Now it follows that $f^{(2)}$ is Lipschitz on X with respect to dis(.,.). Thus, in order to finish the proof of claim 1 we need to show the following claim 2.

CLAIM 2: *For every $\varepsilon > 0$ and $k \in \mathbb{N}$ there is a block basis $(x_i)_{i=1}^{k}$ of (e_i) which is normalized in X, having the property that the set*

$$B^+(x_1, \ldots, x_k) = \left\{\sum_{i=1}^{k} \alpha_i x_i \colon 0 \le \alpha_i \le 1 \ , \ \left\|\sum_{i=1}^{k} \alpha_i x_i\right\| = 1\right\}$$

has diameter less than ε with respect to dis$(\cdot,\cdot)$.

Proof of Claim 2:

CASE 1: $X = \ell_p$, $1 \le p < \infty$. In this case we consider as in [L] the "rationalized" version of ℓ_p, i.e., $\ell_p(D) = \{(x_q)_{q\in D} \colon \sum_{q\in D} |x_q|^p < \infty\}$ where $D = \mathbb{Q} \cap (0,1)$. Let $(e_q)_{q\in D}$ denote the natural basis of $\ell_p(D)$. For $n \in \mathbb{N}$ define the operator $T_n \colon \ell_p(D) \to \ell_p(D)$ by

$$T_n\left(\sum_{q\in D} \alpha_q e_q\right) = \sum_{j=1}^{n} \sum_{q\in D} \alpha_q e_{(q+j-1)/n} \, .$$

For every $n \in \mathbb{N}$, $\lambda_n = n^{1/p}$ is an approximate eigenvalue of T_n [L] and since T_n and T_m commute for $n, m \in \mathbb{N}$ one can choose for a fixed $m \in \mathbb{N}$, $m \gg k$, and $\delta > 0$ a vector $u = \sum_{q\in D} u_q e_q \in \mathrm{Ba}(\ell_p(D))$ so that $\mathrm{supp}(u) = \{q \in D \colon u_q \neq 0\}$ is finite, and so that $\|T_n(u) - n^{1/p} u\| < \delta$ for all $m \le n$.

Let $x_1 < x_2 < \cdots < x_m$ be elements of ℓ_p $(= \ell_p(\mathbb{N}))$, each having the same distribution as u (i.e., $x_k \stackrel{\text{dist}}{=} \sum_{i=1}^{s} u_{q_i} e_i$ where $q_1 < q_2 < \cdots < q_s$ and $\mathrm{supp}(u) = \{q_1, q_2, \ldots, q_s\}$). We deduce that for any scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ with

$\|\sum_{i=1}^{k}\alpha_i e_i\| = 1$ we have

$$\mathrm{dis}\Big(x_1\ ,\ \sum_{i=1}^{k}\alpha_i x_i\Big) \overset{\delta_1}{=} \mathrm{dis}\Big(x_1\ ,\ \sum_{i=1}^{k}\Big(\frac{m_i}{m}\Big)^{1/p} x_i\Big)$$
$$\overset{\delta}{\leq} \mathrm{dis}\Big(\frac{1}{m^{1/p}}\sum_{i=1}^{m} x_i\ ,\ \sum_{i=1}^{k}\Big(\frac{m_i}{m}\Big)^{1/p} x_i\Big)$$
$$\leq \frac{1}{m^{1/p}}\sum_{i=1}^{k}\mathrm{dis}\Big(\sum_{j=1}^{m_i} x_j\ ,\ (m_i)^{1/p} x_i\Big) \leq k\cdot\delta\ ,$$

where $m_1,\ldots,m_k \in \mathbb{N}$ with $\sum_{i=1}^{k} m_i = m$ are chosen so that $\sum|\alpha_i - (\frac{m_i}{m})^{1/p}|$ is minimal, and where δ_1 depends on m and decreases to zero for $m \to \infty$. Thus, choosing m big enough and δ small enough we deduce claim 2 in the case that $X = \ell_p$.

CASE 2: $X = c_0$. In this case Lemberg's argument does not work, but we are able to explicitly write down the desired vectors $x_1, x_2, \ldots, x_k$.

For $0 < r < 1$ we will define a sequence of vectors $(y^{(n)}: n \in \mathbb{N}_0)$ in $\mathrm{Ba}(c_0)\cap c_{00}$. We put $y^{(0)} = e_1$ and assuming $y^{(n)} = \sum_{i=1}^{\ell_n} y_i^{(n)} e_i$ is chosen we put

$$y^{(n+1)} = \sum_{i=1}^{\ell_n}\Big(r^{n+1}e_{3(i-1)+1} + y_i^{(n)}e_{3(i-1)+2} + r^{n+1}e_{3(i-1)+3}\Big)$$

(thus $y^{(1)} = (r,1,r,0,\ldots)$, $y^{(2)} = (r^2,r,r^2,r^2,1,r^2,r^2,r,r^2,0,\ldots)$, etc.). Choosing $r < 1$ big enough and $n \in \mathbb{N}$ big enough and letting $x_1 < x_2 < \cdots < x_k$ all have the same distribution as $y^{(n)}$ one also deduces claim 2. ∎

Remark: After we obtained Theorem 9, we learned of the result of T. Gowers [G], which yields a considerably deeper version of Claim 1 in the case $X = c_0$. Gowers showed that for every Lipschitz function $f: c_0 \to \mathbb{R}$ and every $\varepsilon > 0$ there is an infinite dimensional subspace Y of c_0 so that $\mathrm{osc}(f|_{S_Y}) < \varepsilon$. This is false for $X = \ell_p$ $(1 \leq p < \infty)$ [OS]. ∎

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